Chip-firing games on graphs

A. Björner

Department of Mathematics, Royal Institute of Technology,
S-10044 Stockholm, Sweden

L. Lovász

Department of Computer Science, Eötvös Loránd University,
Budapest, Hungary H-1088, and
Department of Computer Science, Princeton University,
Princeton, NJ 08544

P. W. Shor

AT&T Bell Laboratories, Murray Hill, NJ 07974

Abstract

We analyze the following (solitaire) game: each node of a graph contains a pile of chips, and a move consists of selecting a node with at least as many chips on it as its degree, and let it send one chip to each of its neighbors. The game terminates if there is no such node. We show that the finiteness of the game and the terminating configuration are independent of the moves made. If the number of chips is less than the number of edges, the game is always finite. If the number of chips is at least the number of edges, the game can be infinite for an appropriately chosen initial configuration. If the number of chips is more than twice the number of edges minus the number of nodes, then the game is always infinite.

The independence of the finiteness and the terminating position follows from simple but powerful "exchange properties" of the sequences of legal moves, and from some general results on "antimatroids with repetition", i.e., languages having these exchange properties. We relate the number of steps in a finite game to the least positive eigenvalue of the Laplace operator of the graph.

0   Introduction

Studying a certain "balancing game", J. Spencer (1986) introduced the following (solitaire) game. We start with N chips in a pile. We move $\lfloor N/2 \rfloor$ of them to the right and $\lfloor N/2 \rfloor$ of them to the left by one unit (if the number of chips is odd, one chip stays in its original position). In the next step, we do the same with each of the piles etc. Spencer proved that for the first N steps, the procedure is well approximated by the "Galton process", i.e., the procedure in which a fractional number of chips is allowed and the pile of N is split into two piles of size $N/2$.

Inspired by these results, R. Anderson, L. Lovász, P. Shor, J. Spencer, É. Tardos and S. Winograd (1987) examined this procedure in greater detail. In some respects it turned out to be more natural to study the refined procedure in which an elementary step was to select a pile with more that one chip and move one chip of the pile to the left and one to the right. Among other results, it was shown that this procedure terminates in exactly $k(k-1)(2k-1)/6$ steps, where $k = \lfloor \frac{N+2}{2} \rfloor$. Moreover, independently of the selections made during the procedure, the terminating configuration consists of N consecutive single chip piles, if N is odd, and $N+1$ consecutive single chip piles with the middle pile missing, if N is even.

It is natural to consider this procedure on an arbitrary finite graph G. We start with a pile of some chips on each node, N chips altogether. A step consists of selecting a node v which has at least as many chips as its degree, and move one chip from v to each of its neighbors. We call this step *firing* the node v. The game terminates if each node has fewer chips than its degree. The original game corresponds to the case when the graph is a sufficiently long path and all chips are piled up on the middle node.

In general, the finiteness of this procedure will depend on both the graph and the distribution of the chips. However, we shall show that it does not depend on the choices made during the game: the graph and the original position of the chips determines both the number of steps and (if this number is finite) the terminating configuration. It seems that there are various ways to put this fact in a more general context. First, the positions of the game can be viewed as a so-called "Church-Rosser system". Second, the feasible sequences of moves form a language that has strong "exchange prop-

erties". These properties are extensions of exchange properties of greedoids and, more specifically, of antimatroids, to languages that allow repetition. We shall pursue this second approach because it will yield a more thorough understanding of the structure of feasible games. Among others, it follows that the positions that can be reached from a given beginning position form a locally free semimodular lattice. (For more on connections between greedoids and Church-Rosser systems, see also Faigle, Goecke and Schrader (1987)). A third way of putting this game in a more general context is to view it as a Petri net. (For an introduction to Petri nets, see Reisig (1985)). Although we have not been able to use this connection, it is possible that it could also contribute to understanding the game.

Our main concern will be to study the finiteness of the procedure. It is clear that if the number of chips, N, is more than $2 \cdot |E(G)| - |V(G)|$ then the procedure cannot terminate: with so many chips, one node will have at least as many as its degree. If $N \leq 2 \cdot |E(G)| - |V(G)|$ then clearly we can have a terminating position. We shall prove that if the number of chips is less than $|E(G)|$ then the game always terminates; while if $|E(G)| \leq N \leq 2 \cdot |E(G)| - |V(G)|$ then the game may or may not terminate, depending on the original configuration of the chips (both possibilities occurring for every graph and every number of chips in this range).

Our proof gives a rather poor (exponential) bound on the number of steps in which the procedure terminates (if it terminates at all). G. Tardos (1987) proved that if the procedure terminates than it terminates in $O(|V(G)|^4)$ steps. In section 3, we relate the chip-firing game to the eigenvalues of the Laplace matrix of the graph. This approach yields a bound on the length of this procedure in terms of the smallest positive eigenvalue of the Laplace matrix. This bound is, in terms of the number of nodes of the graph, weaker then the bound given by Tardos, but if the least positive eigenvalue of the Laplace matrix is not too small (the graph has good expanding properties), then it may be better.

1 The chip firing game and greedoids with repetition

Let us fix a connected graph G (without loops or multiple edges) on node set $1, \ldots, n$, and start by putting a_i chips on node i, $i = 1, \ldots, n$. So $a \in Z_+^n$ and $\sum_i a_i = N$. Recall that *firing* the node i means that we decrease a_i by the degree $\deg(i)$ of node i, and increase a_j by 1 for each neighbor j of i. Formally, we can define the vector w_i by

$$(w_i)_j = \begin{cases} \deg(i), & \text{if } j = i, \\ -1, & \text{if } ij \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Then firing i means subtracting w_i from a; this step is legal if $a - w_i \geq 0$.

A *position* in the game is any distribution of the chips on the graph, i.e., any vector $b \in Z_+^n$ such that $\sum_i b_i = N$. A *legal game* is any sequence of positions, starting with a, such that each position is obtained from the previous by legal step.

Let us record the order in which the nodes are fired in a game; this gives us a word formed from the nodes as letters. Let $\mathcal{L}$ denote the set of records of all legal games. We shall show that $\mathcal{L}$ has special exchange properties which imply that if it is finite then it is an "antimatroid with repetition" (see below for definition). This will yield the following theorem:

1.1 Theorem. *Given a connected graph and an initial distribution of chips, either every legal game can be continued indefinitely, or every legal game terminates after the same number of moves with the same final position. The number of times a given node is fired is the same in every legal game.*

Let E be a finite set and $\mathcal{L}$, a language over E, i.e., a set of finite strings formed from the elements of E. A *subword* of a word α is obtained by deleting letters from α arbitrarily (so a subword need not consist of consecutive letters of α). We denote by $|\alpha|$ the length of the word α, and by $[\alpha]$, the "score" of the word α, i.e., the vector in Z_+^n defined by

$[\alpha]_i = k$, if i occurs k times in α.

For two vectors $u, v \in R^n$, we shall denote by $u \vee v$ their (coordinate-wise) maximum and by $u \wedge v$, their coordinate-wise minimum. We shall denote by $|u|_1$ the l_1-norm (sum of absolute values of entries) of the vector u.

We say that the language $\mathcal{L}$ is *left-hereditary*, or briefly *hereditary*, if whenever a string belongs to it, every beginning section of this string also belongs to it. We say that $\mathcal{L}$ is *locally free* if the following holds:

(LF) Let $\alpha \in \mathcal{L}$ and $x \neq y$, two letters in E such that $\alpha x \in \mathcal{L}$, and $\alpha y \in \mathcal{L}$. Then $\alpha xy \in \mathcal{L}$.

We say that $\mathcal{L}$ is *permutable* if:

(PM) Whenever $\alpha, \beta \in \mathcal{L}$, $[\alpha] = [\beta]$ and $\alpha x \in \mathcal{L}$ for some $x \in E$, we also have $\beta x \in \mathcal{L}$.

It will be easy to verify that the records of legal games have these properties. On the other hand, just these properties of a language have quite strong implications. The key to these is the following "strong" exchange property:

(SE) If $\alpha, \beta \in \mathcal{L}$ then α contains a subword α' such that $\beta \alpha' \in \mathcal{L}$ and $[\beta \alpha'] = [\alpha] \vee [\beta]$.

This last property is a very strong version of the *greedoid exchange property*:

(GE) If $\alpha, \beta \in \mathcal{L}$ and $|\beta| < |\alpha|$ then there exists a letter x in α such that $\beta x \in \mathcal{L}$.

If $\mathcal{L}$ is *simple*, i.e., no word in $\mathcal{L}$ contains the same letter more than once, then (GE) defines *greedoids* (among all left-hereditary languages). Since this case does not concern us in this paper, we refer the reader to Korte and Lovász (1983) for other definitions of greedoids and some basic examples and properties. In the case of simple languages, (SE) defines a special class of greedoids, called *antimatroids*. Antimatroids were introduced by Edelman (1980) and Jamison (1982) as closure operations abstracting the combinatorial properties of convexity and (equivalently) as locally free semimodular lattices; as special left-hereditary languages they were characterized by Korte and Lovász (1984). The characterization as left-hereditary simple languages with the strong exchange property was given by Björner (1985). So the languages considered in this paper may be viewed as "antimatroids with repetition". Björner (1985) and Björner and Ziegler (1987) extended the notion of greedoids to languages with repeated letters. We refer to the latter for a

discussion of other related exchange properties.

1.2 Lemma. *Every locally free permutable left-hereditary language has the strong exchange property. Conversely, every language with the strong exchange property is locally free and permutable.*

Proof. We use induction on $|[\alpha] \vee [\beta]|_1$. Let α' consist of those letters x in α which are preceded by at least $[\beta]_x$ occurrences of x. We show that $\beta\alpha' \in \mathcal{L}$; the fact that $[\beta\alpha'] = [\alpha] \vee [\beta]$ is obvious.

Let α'' be the longest prefix of α' such that $\beta\alpha'' \in \mathcal{L}$. Assume, by way of contradiction, that $\alpha'' \neq \alpha'$, and let x be the letter in α' following α''. The fact that x occurs in α' at all implies that

$$[\alpha]_x = [\beta]_x + [\alpha']_x > [\beta]_x + [\alpha'']_x = [\beta\alpha'']_x,$$

and hence we can write α as $\alpha_1 x \alpha_2$ where $[\alpha_1]_x = [\beta\alpha'']_x$.

Now $[\alpha_1] \leq [\beta\alpha'']$, and hence $[\alpha_1] \vee [\beta\alpha''] = [\beta\alpha'']$. Moreover, $|[\beta\alpha'']|_1 < |[\beta\alpha']|_1 = |[\alpha] \vee [\beta]|_1$, and so we can apply the induction hypothesis and find a subword γ of $\beta\alpha''$ such that $\alpha_1 \gamma \in \mathcal{L}$ and $[\alpha_1 \gamma] = [\beta\alpha'']$. Clearly x does not occur in γ. Since both $\alpha_1 x$ and $\alpha_1 \gamma$ are in $\mathcal{L}$, we can apply (LF) repeatedly and obtain that $\alpha_1 \gamma x \in \mathcal{L}$. Now (PM) implies that $\beta\alpha'' x \in \mathcal{L}$, which contradicts the choice of α''.

The converse, which is more straightforward, is left to the reader.

∎

Let $\mathcal{L}$ be any left-hereditary language. We say that a word $\alpha \in \mathcal{L}$ is *basic*, if it is not a proper beginning section of any word in the language. It is clear that if a left-hereditary language has the strong exchange property (or just the greedoid exchange property) and has a basic word, then every basic word has the same length, and no word is longer. We call the common length of basic words the *rank* of the language. We say that the rank is infinite if there are no basic words. Note that in this latter case, every word can be extended indefinitely.

The strong exchange property also implies that if α and β are basic words then $[\alpha] = [\beta]$. We can generalize this observation as follows. We say that two words α and β in a left-hereditary language $\mathcal{L}$ are *equivalent* (in notation $\alpha \sim \beta$), if for every string γ, $\alpha\gamma \in \mathcal{L}$ iff $\beta\gamma \in \mathcal{L}$. The equivalence classes of

this equivalence relation are called *flats*. We say that a flat f is a *subflat* of flat g, if every word in f can be extended to a word in g. Note that it would be enough to require this for a single word in f: if $\alpha \in f$ and $\alpha\gamma \in g$, and β is equivalent to α, then $\beta\gamma$ is in $\mathcal{L}$ by the definition of equivalence, and $\beta\gamma$ is trivially equivalent to $\alpha\gamma$. For a study of flats in general greedoids, see Björner (1985), Crapo (1984), and Björner, Korte and Lovász (1985).

We show now that the flats of a left-hereditary language with the strong exchange property have a particularly simple structure:

1.3 Lemma. *Let $\mathcal{L}$ be a locally free permutable left-hereditary language of finite rank and $\alpha, \beta \in \mathcal{L}$. Then the flat defined by α is a subflat of the flat defined by β iff $[\alpha] \leq [\beta]$. In particular, $\alpha \sim \beta$ iff $[\alpha] = [\beta]$.*

Proof. Assume that $[\alpha] \leq [\beta]$. Then by the strong exchange property, there exists a word γ such that $\alpha\gamma \in \mathcal{L}$ and $[\alpha\gamma] = [\beta]$. By permutability, $\alpha\gamma$ belongs to the same flat as β, and so α belongs to a subflat of this.

Conversely, assume that α defines a subflat of the flat defined by β, and let $\alpha\gamma$ be an extension of α in this flat. Let $\alpha\gamma\delta$ be an extension to a basic word. Then $\beta\delta$ is also in the language, and is clearly also basic. But then, as remarked, $[\alpha\gamma\delta] = [\beta\delta]$ and hence $[\alpha] \leq [\beta]$.

∎

So the flats of such a language can be identified with the scores of words. These scores are partially ordered (by the coordinate-wise ordering) and form a lattice (since by the strong exchange property, the coordinate-wise maximum of two scores is again a score). It follows easily from the local freeness of the language that this lattice is locally free (i.e., the interval from any lattice element x to the join of all elements covering x is a boolean algebra; see Edelman (1980) and Crapo (1984)). This property implies that the lattice is semimodular (i.e., if x covers $x \wedge y$ then $x \vee y$ covers y). The lattice rank of a score vector equals its l_1-norm.

Now we apply these general results to the language of legal games on a given graph, starting from a given position.

1.4 Lemma. *The records of legal games form a locally free permutable left-hereditary language.*

Proof. It is trivial that this language is left-hereditary. To check property (LF), note that if node y can be fired after the game α, i.e., it has enough chips on it, then this remains true after the firing of node x, (this can only increase the number of chips on y), i.e., αxy is a legal game. To show that the language is permutable, note that if the games α and β have the same score then they lead to the same position, and hence any legal continuation of one is also a legal continuation of the other.

∎

The positions in the game have a very simple interpretation in terms of the language.

1.5 Lemma. *Suppose the language of legal games is of finite rank. Then two legal games lead to the same position if and only if they have the same score. (Hence, the positions in the game can be identified with the flats of the language.)*

Proof. Let α and β lead to the same position. Then they have the same continuations, and hence $\alpha \sim \beta$. By Lemma 1.3, this implies that $[\alpha] = [\beta]$. Conversely, if $[\alpha] = [\beta]$ then, as already used, they lead to the same position.

∎

Now the proof of Theorem 1.1 is immediate: terminating legal games correspond to basic words, and so by the above, they have the same length and same score, and lead to the same position.

1.6 Remark. The argument of Lemma 1.4 carries over to the following more general "game": let $Ax \leq b$ ($b \geq 0$) define a non-empty polyhedron P in R^d. Let E be any finite set of vectors in R^d with the property that for each row vector a of A, at most one of the inner products ax ($x \in E$) is positive. Consider the language of those sequences $x_1 x_2 \ldots x_n$, $x_i \in E$, which satisfy $x_1 + \ldots + x_j \in P$ for all $1 \leq j \leq n$. Then this language is left-hereditary, locally free and permutable.

2 The finiteness of the game

In this section we study the question of which chip firing games are infinite and which are finite. Let G be a connected graph with n nodes and m edges, and let us have N chips. We start with a simple lemma.

2.1 Lemma. *If a chip-firing game is infinite then every node is fired infinitely often.*

Proof. There is a node v which is fired infinitely often. Let u be any neighbor of v. Then every time v is fired, it sends a chip to u. Since u cannot compile more than N chips, it must be fired infinitely often itself. Since the graph is connected, this means that every node is fired infinitely often.

∎

Let us contrast this with the following fact proved by G. Tardos (1987):

2.2 Lemma. *If the chip-firing game terminates then there is a node which is not fired at all.*

Proof. For sake of completeness, we give Tardos's simple proof. We want to show that once every node has been fired, the game cannot get stuck. Consider the node v that has been idle for the longest time. Then all of its neighbors have been fired since the last firing of v, and since it received a chip from each of its neighbors, v must have at least $\deg(v)$ chips. So v can be fired.

∎

The main result in this section relates the finiteness of the game to the number of chips.

2.3 Theorem.

(a) *If $N > 2m - n$ then the game is infinite.*

(b) *If $m \leq N \leq 2m - n$ then there exists an initial configuration guaranteeing finite termination and also one guaranteeing infinite game.*

(c) *If $N < m$ then the game is finite.*

Proof. It is obvious that if $N > 2m - n$ then the game cannot terminate: there is always a node v with at least $\deg(v)$ chips on it. It is also obvious that if $N \leq 2m - n$ then we can place at most $\deg(v) - 1$ chips on each node v, and so there are configurations with no legal move.

Next we show that if the number of chips is at least m then there is an initial configuration that leads to an infinite game. Clearly it suffices to show this for $N = m$. Consider any acyclic orientation of G, and let $\deg^+(v)$ denote the out-degree of node v. Let us place $\deg^+(v)$ chips on each node v; this is clearly possible since there are m chips altogether. We claim that this game is infinite.

Observe first that there must exist a node that can be fired in the first step. In fact, the orientation is acyclic, which implies that there is a source, i.e., a node v with $\deg(v) = \deg^+(v)$. Now fire this node, and observe that the resulting distribution of chips can also be obtained from an orientation: if we reverse the edges incident with v, we decrease the outdegree of v by $\deg^+(v)$, and increase the outdegree of each of its neighbors by 1. Since reversing the edges incident with a source does not create any directed cycle, we can find a source in the resulting digraph which can again be fired, etc.

The proof of (c) is motivated by the previous construction. Consider any distribution of $N < m$ chips on the nodes; let $f(v)$ denote the number of chips on node v. Also consider an acyclic orientation of the graph G and the quantity

$$T = \sum_{v \in V(G)} \max\{0, f(v) - \deg^+(v)\}.$$

We say that a node u is *deficient* if $f(u) < \deg^+(u)$; by our hypothesis that $N < m$, there must exist a deficient node. We are going to show that we can modify the orientation during the game so that T never increases and if the set of deficient nodes changes then T must actually decrease. If the game is infinite then every node gets fired infinitely often, and hence the set of deficient nodes must change infinitely often (since a deficient node cannot be fired). Since T cannot decrease infinitely often, this implies that the game is finite.

Consider the node v that is first fired; we have $f(v) \geq \deg(v)$. Fire v and reverse the orientation of all edges leaving v. We do not create any

cycle. Moreover, we do not increase T since the term in T corresponding to v decreases by $\deg(v) - \deg^+(v)$ while each of the $\deg(v) - \deg^+(v)$ terms corresponding to the nodes u for which $uv \in E(G)$ increases by at most 1. Also note that if such a node u was deficient then T actually decreases; if none of these was deficient then the set of deficient nodes did not change. As remarked, this proves the theorem.

■

2.4 Remark. Suppose the chip-firing game is played on a directed graph with n nodes and m arcs, so that when vertex v is fired then $\deg^+(v)$ chips are moved from it along the outgoing arcs to the neighbors dominated by v. Suppose furthermore that the graph has no sinks, so that every node has an arc v leaving it. The same analysis as in Lemma 1.4 shows that the language of legal games is left-hereditary, locally free and permutable (see also Remark 1.6). So Theorem 1.1 is valid also for this directed version, and one can ask for a characterization of those digraphs and initial chip configurations that guarantee finite termination. Clearly, if $N > m - n$ then the game is infinite (there must be a node v with at least $\deg^+(v)$ chips on it). For $N \leq m - n$ the situation is more complicated, and we leave open the problem of whether there is some general result for the directed games of which parts (b) and (c) of Theorem 2.3 are special cases.

3   Chips and the eigenvalues of the Laplace matrix

The *Laplace matrix* $L = L_G = (l_{ij})$ of the graph G with n nodes is the $n \times n$ matrix whose rows and columns are indexed by the nodes and

$$
l_{ij} = \begin{cases} \deg(i), & \text{if } i = j; \\ -1, & \text{if } ij \in E(G); \\ 0, & \text{otherwise.} \end{cases}
$$

This matrix is symmetric and clearly 0 is an eigenvalue of it (with corresponding eigenvector $(1, \ldots, 1)^\mathsf{T}$). Moreover, L is positive semidefinite, which

is easily seen by expressing the corresponding quadratic form as a sum of squares:

$$x^\mathsf{T} L x = \sum_{ij \in E(G)} (x_i - x_j)^2.$$

If G is connected (which we assume throughout this paper), then all the other eigenvalues of L are positive, since the above quadratic form is clearly non-zero unless all entries of x are the same. The least of these non-zero eigenvalues, which we shall denote by λ_1, is closely related to several "expanding" properties of G (see Alon 1986). We shall need the following (probably folklore) lower bound on it (cf. also Alon 1986). Let d denote the diameter of G.

3.1 Lemma. $\lambda_1 \geq 1/(nd)$.

Proof. Let w be an eigenvector belonging to λ_1 with unit length. Then there is a node, say node 1, such that $|w_1| \geq 1/\sqrt{n}$. We may assume without loss of generality that $w_1 > 0$. Since w is orthogonal to the eigenvector belonging to the eigenvector 0, we have $\sum_j w_j = 0$, and hence there exists a k such that $w_k < 0$. Let, say, nodes 1, 2, ..., k form a minimal path from 1 to k (so that $k - 1 \leq d$ in this labelling).

Now we have

$$
\begin{aligned}
\lambda_1 &= w^\mathsf{T} L w \\
&= \sum_{ij \in E(G)} (w_i - w_j)^2 \geq \sum_{i=1}^{k-1} (w_i - w_{i+1})^2 \\
&\geq \frac{1}{k-1} \left(\sum_{i=1}^{k-1} (w_i - w_{i+1}) \right)^2 \\
&= \frac{1}{k-1} (w_1 - w_k)^2 \geq \frac{1}{(k-1)n} \geq \frac{1}{nd} \ .
\end{aligned}
$$

Let us return to the chip firing game. We prove the following bound on the length of the game:

3.2 Theorem. The number of steps in any terminating chip-firing game with N chips is at most $2nN/\lambda_1$.

Proof. Assume that we started with a_i chips on node i and after a total of s steps, we have b_i chips on node i. Let x_i denote the number of times node i was fired. By the lemma of Tardos mentioned in the previous section, if the game is finite then there is a node, say node n, such that $x_n = 0$.

Straightforward counting yields the formula

$$Lx = a - b.$$

We can express L in the following form:

$$L = \sum_{i=1}^{n-1} \lambda_i \cdot v_i v_i^\mathsf{T},$$

where $\lambda_1 \leq \ldots \leq \lambda_{n-1}$ are the non-zero eigenvalues of L and $v_1, \ldots, v_{n-1}$ are corresponding orthogonal eigenvectors of unit length. Consider the generalized inverse L' of L, defined by

$$L' = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i v_i^\mathsf{T}.$$

Then, setting $v_n = \frac{1}{\sqrt{n}}\mathbf{1}$, we get by straightforward computation

$$L'L = \sum_{i=1}^{n-1} v_i v_i^\mathsf{T} = I - v_n v_n^\mathsf{T} = I - \frac{1}{n}J,$$

where I is the identity matrix and J is the all-1 matrix (since $\{v_1, \ldots, v_n\}$ is an orthonormal basis). Hence

$$e_n^\mathsf{T} L'L = e_n^\mathsf{T} - \frac{1}{n}\mathbf{1}^\mathsf{T},$$

and

$$s \;\; = \;\; \mathbf{1}^\mathsf{T} x = (n e_n^\mathsf{T} - n e_n^\mathsf{T} L'L)x = -n e_n^\mathsf{T} L'Lx$$

12

$$= -n e_n^\mathsf{T} L'(a-b) = \sum_{i=1}^{n-1} -\frac{n}{\lambda_i}(e_n^\mathsf{T} v_i)(v_i^\mathsf{T}(a-b)).$$

So we can estimate s as follows (using the Cauchy-Schwartz inequality):

$$
\begin{aligned}
s &\leq \frac{n}{\lambda_1}\sum_{i=1}^{n-1}|e_n^\mathsf{T} v_i| \cdot |v_i^\mathsf{T}(a-b)| \\
&\leq \frac{n}{\lambda_1}\sqrt{\sum_{i=1}^{n-1}(e_n^\mathsf{T} v_i)^2}\sqrt{\sum_{i=1}^{n-1}(v_i^\mathsf{T}(a-b))^2} \\
&\leq \frac{n}{\lambda_1}|a-b| \leq \frac{2nN}{\lambda_1}.
\end{aligned}
$$

■

Using Lemma 3.1, we get that $s < 2n^2 dN < 2n^3 N$. This is slightly worse than the result of Tardos mentioned in the Introduction, since N may be as large as $\Omega(n^2)$. On the other hand, for "expanding" graphs λ_1 is bounded from below by a constant, so in this case our bound is tighter.

References

N. Alon (1986), Eigenvalues and expanders, *Combinatorica* 6, 83-96.

R. J. Anderson, L. Lovász, P. W. Shor, J. Spencer, É. Tardos, and S. Winograd (1987), Disks, Balls, and Walls: Analysis of a combinatorial game, to appear in *Amer. Math. Monthly*).

A. Björner (1985), On matroids, groups, and exchange languages, in *Matroid Theory and its Applications* (ed. L. Lovász and A. Recski), Coll. Math. Soc. J. Bolyai 40; North-Holland, Amsterdam; 25-60.

A. Björner, B. Korte and L. Lovász (1985), Homotopy properties of greedoids, *Advances in Appl. Math.* 6, 447-494.

A. Björner and G. M. Ziegler (1987), Introduction to greedoids, to appear in: *Combinatorial Geometries, Advanced Theory* (N. White, ed.), Cambridge Univ. Press.

H. Crapo (1984), Selectors. A theory of formal languages, semimodular lattices branchings, and shelling processes, *Advances in Math.* 54, 233-277.

P. Edelman (1980), Meet-distributive lattices and the anti-exchange closure, *Alg. Univ.* 10, 290-299.

U. Faigle, O. Goecke and R. Schrader (1987), Church-Rosser decomposition in combinatorial structures, preprint, Inst. f. Operations Research, Univ. Bonn.

R. E. Jamison (1982), A perspective on abstract convexity: classifying alignments by varieties, in *Convexity and Related Combinatorial Geometry* (ed. D. C. Kay and M. Breehm), 113-150.

B. Korte and L. Lovász (1983), Structural properties of greedoids, *Combinatorica* 3, 359-374.

B. Korte and L. Lovász (1984), Shelling structures, convexity, and a happy end, in *Graph Theory and Combinatorics*, Proc. Cambridge Combinatorial Conference in Honour of P. Erdős (ed. B. Bollobàs), Academic Press, New York/London, 219-232.

W. Reisig (1985), *Petri Nets: An Introduction*, Springer-Verlag, New York.

J. Spencer (1986), Balancing vectors in the max norm, *Combinatorica* 6, 55-66.

G. Tardos (1987), Polynomial bound for a chip firing game on graphs (preprint).